On images of linear maps with skew derivations

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- For **additive subgroups** A, B of R, let

$$[A, B]$$

denote the additive subgroup of R generated by all elements $[a, b]$ for $a \in A$ and $b \in B$.

Definition

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A derivation of R is called ***outer*** if it is not inner.

Definition

*Let C be a field and A be a C-algebra. An additive map $f : A \to A$ is called C-**linear map** if*

$$f(\beta x) = \beta f(x)$$

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Corollary

(Eroğlu and Lee, 2017)

*δ is **inner** if and only if $\delta(R) \subseteq [R, R]$.*

QUESTION

Characterize derivations δ of R and positive integers n such that

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Question 1. In case R is a prime ring with a nonzero derivation δ and Martindale symmetric ring of quotients Q, characterize

$$\phi(x) = \sum_{i,j} a_{ij}\delta^j(x)b_{ij}$$

for $x \in R$ where a_{ij}, b_{ij} are finitely many elements in Q such that

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Q is also a prime ring and C is a field.

It is known that any derivation $\delta : R \to R$ can be uniquely extended to a derivation of Q, denoted by δ also.

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Let

$$Q[t;\delta] := \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \ldots, a_n \in Q, n \geq 0\},$$

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Question 2. In case R is a prime ring with a nonzero derivation δ and Martindale symmetric ring of quotients Q, characterize

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such that $f(\delta)(R) \subseteq [R,R]$.

Definition

Let R_F *be the Martindale left ring of quotients of* R*. A derivation* $\delta\colon R \to R$ *is called* ***quasi-algebraic*** *if there exist* $b_1, \ldots, b_{n-1}, b \in R_F$ *such that*

$$\delta^n(x) + b_1\delta^{n-1}(x) + \cdots + b_{n-1}\delta(x) = bx - xb \tag{1}$$

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The least integer n is called ***quasi-algebraic degree*** of δ and is denoted by $out - deg(\delta)$.

$out - deg(\delta) = 1$ if and only if δ is X-inner.

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$$p(t) = t^{p^s} + \alpha_1 t^{p^{s-1}} + \cdots + \alpha_s t \text{ if } \operatorname{char} R = p > 0 \text{ and } R \text{ satisfies (1)}$$

where $\alpha_i \in C$.

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In either case, $p(t)$ is called the ***associated polynomial*** of δ.

Note that $p(\delta) = \operatorname{ad}(b)$ for some $b \in Q$.

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such that $\phi(R) \subseteq [R,R]$ and characterize

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n \in Q[t;\delta]$$

such that $f(\delta)(R) \subseteq [R,R]$.

Lemma

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$f(\delta)(R) \subseteq [R, R]$ *if and only if* δ *is quasi-algebraic and* $p(t)|f(t)$.

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$f(\delta)(R) \subseteq [R, R]$ if and only if δ is quasi-algebraic and $p(t)|f(t)$.

For $A(t)$, $B(t) \in Q[t; \delta]$ with $A(t) \neq 0$ by $A(t)|B(t)$ we mean there exists some $q(t) \in Q[t; \delta]$ such that $B(t) = A(t)q(t)$.

The answer to Question 1 is as follows:

Theorem

(Eroğlu and Lee, 2017) *Let R be a simple GPI-ring with a nonzero derivation δ. Suppose that*

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$\phi(R) \subseteq [R, R]$ *if and only if either* $\sum_{i,j} b_{ij}a_{ij}t^j = 0$ *or* δ *is quasi-algebraic and* $p(t) | \sum_j \left(\sum_i b_{ij}a_{ij} \right) t^j$.

Corollary

(Eroğlu and Lee, 2017) *Let R be a simple GPI-ring with a nonzero derivation δ. Given a positive integer n,*

$$\delta^n(R) \subseteq [R, R] \text{ if and only if } \delta^\ell \text{ is X-inner for some } \ell \leq n.$$

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Corollary

(Eroğlu and Lee, 2017) *Let R be a simple GPI-ring with a nonzero derivation δ. Suppose that $f(\delta)$ is a derivation of R. Then*

$$f(\delta) \text{ is X-inner if and only if } \delta \text{ is quasi-algebraic and } p(t) | f(t).$$

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Let σ be an automorphism of R. An additive map $D : R \to R$ is called a **σ-derivation** *(or a skew derivation) of R if*

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If $\sigma \neq id$, then the map $\sigma - id$ is a σ-derivation.

For $b \in Q$,

$$D(x) = bx - \sigma(x)b$$

is a σ-derivation and it is called an **inner** σ**-derivation** of R.

THANKS FOR ATTENDING :)

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